Modeling phonological processes with recursive program schemes

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Overview

- **Recursive program schemes (RSs)** study structure and complexity of algorithms (Moschovakis, 2019)
- We present **boolean monadic RS (BMRS)** phonological grammars that
  - define a *hierarchy* of local licensing and blocking structures;
  - directly capture *do X unless Y*-type behavior;
  - intensionally express phonologically significant generalizations;
  - are connected to results on computational complexity and learnability (Heinz, 2018);
  - capture both input and output-based mappings, including opacity

Overview

- BMRS provide a glimpse into
  - The *combined map* as a function (available to OT, not to SPE)
  - *Individual* functions which interact (available to SPE, not to OT)
- BMRS offer a framework for describing **operations** (like composition) over individual functions
  - More intuitive than finite-state and logical formalisms

BMRSs: Definition

- An input string is a set of elements $\{1, 2, ..., n\}$
  - ordered by **predecessor function** p, **successor function** s
  - having some **(input) boolean functions** $P(x)$

	$\#$	σ	$\acute{\sigma}$	σ	σ	σ	$\#$
	1	2	3	4	5	6	7
$p(x)$		1	2	3	4	5	6
$s(x)$	2	3	4	5	6	7	
$\#(x)$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$
$\sigma(x)$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
$\acute{\Box}(x)$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$

BMRSs: Definition

- Output string defined by **output boolean functions** $O(x)$

$$\begin{aligned} \#_o(x) &= ? \\ \sigma_o(x) &= ? \\ \acute{\square}_o(x) &= ? \end{aligned}$$

#	σ	σ́	σ	σ	σ	#
1	2	3	4	5	6	7
			↓			
?	?	?	?	?	?	?
1′	2′	3′	4′	5′	6′	7′

(This follows Courcelle 1994; Engelfriet and Hoogeboom 2001)

BMRSs: Definition

- Output string defined by **output boolean functions** $O(x)$

$$
\begin{aligned}
\#_o(x) &= \#(x) \\
\sigma_o(x) &= \sigma(x) \\
\acute{\square}_o(x) &= \acute{\square}(x)
\end{aligned}
$$

#	σ	σ́	σ	σ	σ	#
1	2	3	4	5	6	7
↓						
#	σ	σ́	σ	σ	σ	#
1′	2′	3′	4′	5′	6′	7′

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BMRSs: Definition

Logical syntax

- **terms**

$$T \rightarrow x \mid p(T) \mid s(T)$$

$$x,\ p(x),\ s(s(x)),\ p(p(p(x))),\ \ldots$$

- **boolean expressions**

$$E \rightarrow \top \mid \bot \mid P(T) \mid \texttt{if } E \texttt{ then } E \texttt{ else } E$$

BMRSs: Definition

$$T \rightarrow x \mid p(T) \mid s(T)$$
$$E \rightarrow \top \mid \bot \mid P(T) \mid \texttt{if } E \texttt{ then } E \texttt{ else } E$$

$$\text{final}(x) = \texttt{if } \#(s(x)) \texttt{ then } \top \texttt{ else } \bot$$

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$\sigma(x)$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
$\acute{\square}(x)$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$

BMRSs: Definition

$$T \to x \mid p(T) \mid s(T)$$
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$\acute{\square}(x)$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$
$\#(s(x))$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$
$\text{final}(x)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$

BMRSs: Definition

- We can define the output boolean functions with a **BMRS system of equations**

$$
\begin{aligned}
O_1(x) &= E_1 \\
O_2(x) &= E_2 \\
&\dots \\
O_n(x) &= E_n
\end{aligned}
$$

BMRSs: Definition

$$
\begin{aligned}
\#_{\text{o}}(x) &= \#(x) \\
\sigma_{\text{o}}(x) &= \sigma(x) \\
\acute{\square}_{\text{o}}(x) &= \texttt{if } \mathrm{final}(x) \texttt{ then } \bot \texttt{ else} \\
& \quad \texttt{if } \acute{\square}_{\text{o}}(p(x)) \texttt{ then } \top \texttt{ else} \\
& \quad \acute{\square}(x)
\end{aligned}
$$

BMRSs: Definition

$$\begin{aligned}
\#_o(x) &= \#(x) \\
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\acute{\Box}_o(x) &= \texttt{if}\ \text{final}(x)\ \texttt{then}\ \bot\ \texttt{else} \\
&\quad\ \texttt{if}\ \acute{\Box}_o(p(x))\ \texttt{then}\ \top\ \texttt{else} \\
&\quad\ \acute{\Box}(x)
\end{aligned}$$

	$\#$	σ	$\acute{\sigma}$	σ	σ	σ	$\#$
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$\acute{\Box}_o(x)$							

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	$\#$	σ	$\acute{\sigma}$	σ	σ	σ	$\#$
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&\quad\; \acute{\Box}(x)
\end{aligned}
$$

	#	σ	σ́	σ	σ	σ	#
	1	2	3	4	5	6	7
$\#_o(x)$	⊤	⊥	⊥	⊥	⊥	⊥	⊤
$\sigma_o(x)$	⊥	⊤	⊤	⊤	⊤	⊤	⊥
$\acute{\Box}_o(x)$	⊥	⊥	⊤	⊤	⊤	⊥	⊥
	1′	2′	3′	4′	5′	6′	7′
	#	σ	σ́	σ́	σ́	σ	#

BMRSs: Definition

- BMRS systems of equations always have a *least-fixed point solution* (Moschovakis, 2019)
- If restricted to recursing on only $p(x)$ or $s(x)$ (but not both), BMRSs describe *subsequential functions* (Bhaskar et al., ms)
- The syntax expresses a **hierarchy** of **blocking structures** and **licensing structures**

$$\acute{\square}_o(x) = \texttt{if}\ \mathrm{final}(x)\ \texttt{then}\ \bot\ \texttt{else}\ \texttt{if}\ \acute{\square}_o(p(x))\ \texttt{then}\ \top\ \texttt{else}\ \acute{\square}(x)$$

BMRSs: Input/Output-based mappings

- Input-based: output boolean functions defined **without recursion**
  - Compute output by reference to input structure only
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  - Inventory: H(igh), R(ising), L(ow), F(alling)
  - RR $\rightarrow$ HR (simultaneous, ISL); RRR $\rightarrow$ HHR

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  - RR → HR (simultaneous, ISL); RRR → HHR

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\begin{aligned}
H_o(x) &= \texttt{if } \underline{R}R(x) \texttt{ then } \top \texttt{ else } H(x) \\
R_o(x) &= \texttt{if } \underline{R}R(x) \texttt{ then } \bot \texttt{ else } R(x) \\
L_o(x) &= L(x) \\
F_o(x) &= F(x)
\end{aligned}
$$

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L_o(x) &= L(x)\\
F_o(x) &= F(x)
\end{aligned}
$$

	#	R	R	R	R	#
	1	2	3	4	5	6
$H_o(x)$	⊥	⊤	⊤	⊤	⊥	⊥
$R_o(x)$	⊥	⊥	⊥	⊥	⊤	⊥
	1′	2′	3′	4′	5′	6′
	#	H	H	H	R	#

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  - Refer to **current** input, otherwise to output structure only
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  - LL $\rightarrow$ RL (iterative, ROSL)
  - LLL $\rightarrow$ LRL, LLLL $\rightarrow$ RLRL

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  - LL $\rightarrow$ RL (iterative, ROSL)
  - LLL $\rightarrow$ LRL, LLLL $\rightarrow$ RLRL

$$
\begin{aligned}
R_o(x) &= \texttt{if } \underline{L}L_oR_o(x) \texttt{ then } \top \texttt{ else} \\
&\quad\; \texttt{if } \underline{L}L_o(x) \texttt{ then } \top \texttt{ else} \\
&\quad\; R(x) \\
L_o(x) &= \texttt{if } R_o(x) \texttt{ then } \bot \texttt{ else } L(x) \\
H_o(x) &= H(x) \\
F_o(x) &= F(x)
\end{aligned}
$$

BMRSs: Input/Output-based mappings

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\begin{aligned}
R_o(x) &= \texttt{if } \underline{L}L_oR_o(x) \texttt{ then } \top \texttt{ else} \\
&\quad \texttt{if } \underline{L}L_o(x) \texttt{ then } \top \texttt{ else} \\
&\quad R(x) \\
L_o(x) &= \texttt{if } R_o(x) \texttt{ then } \bot \texttt{ else } L(x)
\end{aligned}
$$

	#	L	L	L	L	#
	1	2	3	4	5	6
$R_o(x)$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$
$L_o(x)$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$
	1′	2′	3′	4′	5′	6′
	#	R	L	R	L	#

BMRSs: Function Composition

- BMRS offers intuitive framework for **function composition**
- Given two BMRS systems of equations a and b, $b \circ a$ is defined:
  - In system b, all non-recursively-defined boolean function names refer to *corresponding* definitions in system a

BMRSs: Function Composition

- BMRS offers intuitive framework for **function composition**
- Given two BMRS systems of equations a and b, $b \circ a$ is defined:
  - In system b, all non-recursively-defined boolean function names refer to *corresponding* definitions in system a
- Applications in phonological process **interactions**
  - Tianjin LL (LL → RL) rule **feeds** RR (RR → HR) rule
  - R<u>LL</u> → <u>R**R**</u>L → **H**RL
  - 'Combined map' (Chandlee, 2019)
  - **Compose** two BMRS systems
    - System a: LL rule
    - System b: RR rule
  - Can do both easily in BMRS formalism

BMRSs: Function Composition

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$$
\begin{aligned}
& \qquad a \\
R_a(x) &= \text{if } \underline{L}L_aR_a(x) \text{ then } \top \text{ else} \\
&\quad\ \text{if } \underline{L}L_a(x) \text{ then } \top \text{ else} \\
&\quad\ R(x) \\
L_a(x) &= \text{if } R_a(x) \text{ then } \bot \text{ else } L(x) \\
H_a(x) &= H(x) \\
F_a(x) &= F(x)
\end{aligned}
$$

BMRSs: Function Composition

a

$$R_a(x) = \text{if } \underline{L}L_aR_a(x) \text{ then } \top \text{ else if } \underline{L}L_a(x) \text{ then } \top \text{ else } R(x)$$

$$L_a(x) = \text{if } R_a(x) \text{ then } \bot \text{ else } L(x)$$

$$H_a(x) = H(x)$$

$$F_a(x) = F(x)$$

b

$$H_b(x) = \text{if } \underline{R}R(x), \text{ then } \top \text{ else H(x)}$$

$$R_b(x) = \text{if } \underline{R}R(x), \text{ then } \bot \text{ else R(x)}$$

$$L_b(x) = L(x)$$

$$F_b(x) = F(x)$$

BMRSs: Function Composition

a

$$
\begin{aligned}
R_a(x) &= \text{if } \underline{L}L_aR_a(x) \text{ then } \top \text{ else} \\
&\quad \text{if } \underline{L}L_a(x) \text{ then } \top \text{ else} \\
&\quad R(x) \\
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H_a(x) &= H(x) \\
F_a(x) &= F(x)
\end{aligned}
$$

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$$
\begin{aligned}
H_b(x) &= \text{if } \underline{R}R(x), \text{ then } \top \text{ else H(x)} \\
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L_b(x) &= L(x) \\
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\end{aligned}
$$

$b \circ a$

$$
\begin{aligned}
H_b(x) &= \texttt{if } \underline{R_a}R_a(x) \texttt{ then } \top \texttt{ else } H_a(x) \\
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BMRSs: Function Composition

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L_b(x) &= L_a(x)\\
F_b(x) &= F_a(x)
\end{aligned}$$

	#	R	L	L	#
$R_b(x)$		$\bot$	$\top$	$\bot$	
$R_a(x)$		$\top$	$\top$	$\bot$	
$H_b(x)$		$\top$	$\bot$	$\bot$	
	#	H	R	L	#

Discussion

- BMRS provide a glimpse into
  - The *combined map* as a function (available to OT, not to SPE)
  - *Individual* functions which interact (available to SPE, not to OT)
  - *Operations* over individual functions (less intuitive in finite-state and logical formalisms)

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- The linchpin: if-then-else syntax
  - Capture *do X unless Y*-type behavior (as in OT)
  - Input- *and* output-orientedness

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    - Further refine definitions
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- The linchpin: if-then-else syntax
  - Capture *do X unless Y*-type behavior (as in OT)
  - Input- *and* output-orientedness
    - Hierarchy of licensing and blocking structures
    - Elsewhere condition

Conclusion

- Express phonologically significant generalizations with BMRS
- Equivalent to subsequential class of functions
- Unique syntax defines hierarchy of local licensing and blocking structures
- Capture input and output-based mappings
- Intuitive framework for examining phonological process interaction

Thank You

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- Identify other operations besides composition
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  - MR rule (MR $\rightarrow$ LR), RM rule (RM $\rightarrow$ HM)
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MR<RM		**RM<MR**	
MRM	RMR	MRM	RMR
LRM	RLR	MHM	HMR
LHM	*RLR	*MHM	HLR

Appendix 1: Other Operations

- Interaction is **ISL** (Oakden and Chandlee, 2019)

$$
\begin{aligned}
L_o(x) &= \texttt{if } \underline{M}R(x) \texttt{ then } \top \texttt{ else } L(x) \\
M_o(x) &= \texttt{if } \underline{M}R(x) \texttt{ then } \bot \texttt{ else } M(x) \\
H_o(x) &= \texttt{if } \underline{R}M(x) \texttt{ then } \top \texttt{ else } H(x) \\
R_o(x) &= \texttt{if } \underline{R}M(x) \texttt{ then } \bot \texttt{ else } R(x) \\
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\end{aligned}
$$

- **Not** a result of *composing* two systems MR and RM
- Composition recreates ordering paradox

Appendix 1: Other Operations

- New operation '$\ominus$'
- Given two BMRS systems of equations a and b, $b \ominus a$ is defined:
  - Identity-map definitions in b are replaced with corresponding *non-identity* definitions in a
  - Otherwise, leave the definition the same

  - Is $\ominus$ just *priority union*? (Kaplan, 1987; Karttunen, 1998)

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  - System a: RM rule
  - System b: MR rule

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- Changting mutual counterbleeding
  - System a: RM rule
  - System b: MR rule
- Corresponds to **simultaneous application**
  - Is $\ominus$ just *priority union*? (Kaplan, 1987; Karttunen, 1998)

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$$
\begin{aligned}
& a \\
L_a(x) &= L(x) \\
M_a(x) &= M(x) \\
H_a(x) &= \text{if } \underline{R}M(x) \text{ then } \top \text{ else } H(x) \\
R_a(x) &= \text{if } \underline{R}M(x) \text{ then } \bot \text{ else } R(x) \\
F_a(x) &= F(x)
\end{aligned}
$$

Appendix 1: Other Operations

$$
\begin{array}{rclrcl}
 & a & & & b & \\
L_a(x) & = & L(x) & L_b(x) & = & \text{if } \underline{M}R(x) \text{ then } \top \text{ else } L(x) \\
M_a(x) & = & M(x) & M_b(x) & = & \text{if } \underline{M}R(x) \text{ then } \bot \text{ else } M(x) \\
H_a(x) & = & \text{if } \underline{R}M(x) \text{ then } \top \text{ else } H(x) & H_b(x) & = & H(x) \\
R_a(x) & = & \text{if } \underline{R}M(x) \text{ then } \bot \text{ else } R(x) & R_b(x) & = & R(x) \\
F_a(x) & = & F(x) & F_b(x) & = & F(x)
\end{array}
$$

Appendix 1: Other Operations

a

$$
\begin{aligned}
L_a(x) &= L(x) \\
M_a(x) &= M(x) \\
H_a(x) &= \text{if } \underline{R}M(x) \text{ then } \top \text{ else } H(x) \\
R_a(x) &= \text{if } \underline{R}M(x) \text{ then } \bot \text{ else } R(x) \\
F_a(x) &= F(x)
\end{aligned}
$$

b

$$
\begin{aligned}
L_b(x) &= \text{if } \underline{M}R(x) \text{ then } \top \text{ else } L(x) \\
M_b(x) &= \text{if } \underline{M}R(x) \text{ then } \bot \text{ else } M(x) \\
H_b(x) &= H(x) \\
R_b(x) &= R(x) \\
F_b(x) &= F(x)
\end{aligned}
$$

$b \ominus a$

$$
\begin{aligned}
L_b(x) &= \texttt{if } \underline{M}R(x) \texttt{ then } \top \texttt{ else } L(x) \\
M_b(x) &= \texttt{if } \underline{M}R(x) \texttt{ then } \bot \texttt{ else } M(x) \\
H_b(x) &= H_a(x) \\
R_b(x) &= R_a(x) \\
F_b(x) &= F(x)
\end{aligned}
$$